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AN INSTABILITY LOCALIZED AT THE INNER

SURFACE OF AN IMPLODING SPHERICAL SHELL

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ABSTRACT

It is shown that in an imploding spherical shell the surface instabilities are of two different types. The first, which occurs at the outer surface, is the Rayleigh-Taylor instability. The second instability occurs at the inner surface. This latter instability is not as disruptive as R-T modes, but it has three basic properties which differ considerably from those of the R-T instability: (1) it is oscillatory at early times; (2) it grows faster in the long wavelength modes; (3) it depends on the equation of state. It is further shown that this new instability is driven by amplified sound waves in the shell.

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BASIC EQUATIONS

We consider an imploding spherical shell that obeys the following ideal fluid equations:

$$\rho\left(\frac{d\underset{\sim}{v}}{dt}\right) = -\underset{\sim}{\nabla} p \ , \tag{1}$$

$$\frac{\partial\rho}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho\underset{\sim}{v}) = 0 \ , \tag{2}$$

$$\frac{d}{dt}(p/\rho^{\gamma}) = 0 \tag{3}$$

A self-consistent description of the shell motion can be obtained by introducing Sedov's hypothesis[8] of self-similar motion, which in the Lagrangian representation is simply given as

$$R(r_o,t) = r_o\ f(t) \ . \tag{4}$$

$$\rho(r_o,t) = \rho_o(r_o)\ f^{-3}, \qquad p(r_o,t) = p_o(r_o)\ f^{-3\gamma} \ . \tag{5}$$

THE UNPERTURBED MOTION

Using the time-dependent pressure in Eq. (1), we obtain the unperturbed motion

$$\ddot{f}(t)f^{3\gamma-2}(t) = -\frac{1}{\rho_o r}\frac{d}{dr}\left(\rho_o^{\gamma}(r)\right) = -\frac{1}{t_c^2} = -1 \ , \tag{6}$$

The time-dependent part of Eq. (6) yields on integration

$$\dot{f} = \left(\frac{2}{\alpha}\left(f^{-\alpha} - 1\right)\right)^{1/2} \ , \tag{7}$$

where the initial values $f(0) = 1$ and $\dot{f}(0) = 0$ have been used. Here $\alpha = 3(\gamma - 1)$. When $\gamma = 5/3$, Eq. (7) can be integrated once again to give

$$f^2 = 1 - t^2 \ , \tag{8}$$

where $0 < t < 1$.

$$\rho_o(r) = \hat{\rho}_o\left((r^2 - r_-^2)/(r_+^2 - r_-^2)\right)^{1/(\gamma - 1)} \ , \tag{9}$$

where r_+ and r_- are the outer and inner radii of the shell;

$$p_o(r) = \hat{p}_o\left((r^2 - r_-^2)/(r_+^2 - r_-^2)\right)^{\gamma/(\gamma - 1)} , \tag{10}$$

where $\hat{p}_o$ is a constant pressure at $r = r_+$.

STABILITY ANALYSIS

When a perturbation $\underset{\sim}{\xi}$ is introduced, the position vector of a fluid element in Lagrangian variables becomes $\underset{\sim}{R} + \underset{\sim}{\xi}$. A straightforward, though somewhat tedious, calculation[7,11] shows that $\underset{\sim}{\xi}$ obeys the equation

$$f^{(3\gamma-1)}\ \ddot{\underset{\sim}{\xi}} = \frac{(\gamma - 1)}{2}(r^2 - r_-^2)\underset{\sim}{\nabla}\sigma + (\gamma - 1)\sigma\ \underset{\sim}{r} + \underset{\sim}{r}\times\ \underset{\sim}{\omega} + (\underset{\sim}{r}\cdot\underset{\sim}{\nabla})\underset{\sim}{\xi}\ , \tag{11}$$

where $\sigma = \underset{\sim}{\nabla}\cdot\underset{\sim}{\xi}$ and $\underset{\sim}{\omega} = \underset{\sim}{\nabla}\times\underset{\sim}{\xi}$.

We now limit the discussion to the case of incompressible, irrotational perturbations for which the surface instabilities occur. That is, we choose the perturbations such that $\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi} = 0$ and $\underset{\sim}{\nabla}\times\underset{\sim}{\xi} = 0$. This implies that $\underset{\sim}{\xi} = \underset{\sim}{\nabla}\chi$ with $\nabla^2\chi = 0$. In spherical coordinates

$$\chi(r,\theta,\phi) = \sum_{\ell m}[\ Q_+^\ell(t)r^\ell + Q_-^\ell(t)r^{-(\ell + 1)}]\ Y_{\ell m}(\theta,\phi)\ . \tag{12}$$

To obtain $Q_\pm^\ell(t)$, we expand the perturbations in spherical harmonics[11]

$$\underset{\sim}{\xi} = \sum_{\ell m}[\ \xi_1^{\ell m}(r,t)\hat{a}_1 + \ \xi_2^{\ell m}(r,t)\hat{a}_2 + \ \xi_3^{\ell m}(r,t)\hat{a}_3]\ , \tag{13}$$

$$\hat{a}_1 = \hat{e}_r Y_{\ell m}\ , \qquad \hat{a}_2 = r\underset{\sim}{\nabla}Y_{\ell m}\ , \qquad \hat{a}_3 = \underset{\sim}{r}\times\underset{\sim}{\nabla}Y_{\ell m}\ . \tag{14}$$

TIME-DEPENDENT EQUATION

The Classical Rayleigh-Taylor Instability:

$$g(t) = \ddot{R}(t) = r\,\ddot{f}(t) = -\,r\,f^{-3\gamma+2}\ , \tag{16}$$

then Eq. (15) can be written as

$$\ddot{Q}^{\ell}_{\pm}(t) + \left[\frac{3}{2} \mp (\ell + \frac{1}{2})\right]\frac{|g(t)|}{R(t)}\,Q^{\ell}_{\pm}(t) = 0\ , \tag{17}$$

Notice that $Q^{\ell}_{-}(t)$ modes localized at the inner surface are oscillatory in the static limit, whereas $Q^{\ell}_{+}(t)$ modes are unstable.

The growth rate of the unstable mode is given by

$$\gamma_t^2 = |g_o|\,(\frac{m}{R_o}) = |g_o|k_\theta\ , \tag{18}$$

This is the growth rate of the classical Rayleigh-Taylor instability for static media.

SOLUTIONS FOR AN ARBITRARY TIME

$$Q^{\ell}_{\pm}(t) = c_1 \mathcal{F}_{\pm}(\ell, \gamma, t) + c_2 \mathcal{G}_{\pm}(\ell, \gamma, t) \ . \tag{19}$$

Here

$$\mathcal{F}_{\pm}(\ell, \gamma, t) = {}_2F_1\left[\ \frac{1}{4} + \frac{2+id_{\pm}}{4\alpha},\ \ \frac{1}{4} + \frac{2-id_{\pm}}{4\alpha},\ \frac{1}{2};\ 1 - f^{-\alpha}\ \right] \tag{20}$$

and

$$\mathcal{G}_{\pm}(\ell, \gamma, t) = \left(1 - f^{-\alpha}\right)^{1/2}\ {}_2F_1\left[\frac{3}{4} + \frac{2+id_{\pm}}{4\alpha},\ \frac{3}{4} + \frac{2-id_{\pm}}{4\alpha},\ \frac{3}{2};\ 1 - f^{-\alpha}\ \right], \tag{21}$$

where $d_{\pm} = \left[8\alpha\{\frac{3}{2} \mp (\ell + \frac{1}{2})\} - (\alpha + 2)^2\right]^{1/2}$.

As in Ref. 10, we obtain the stability criteria by evaluating the asymptotic limits:

$$\lim_{t\to 1} \frac{\mathcal{F}_{\pm}(\ell,\gamma,t)}{f(t)} \simeq a^{\ell}_{\pm}\, f^{(\alpha-2+id_{\pm})/2} + \text{c. c.} \tag{22}$$

and

$$\lim_{t\to 1} \frac{\mathcal{G}_{\pm}(\ell,\gamma,t)}{f(t)} \simeq b^{\ell}_{\pm}\, f^{(\alpha-2+id_{\pm})/2} + \text{c. c.}\,, \tag{23}$$

where

$$a^{\ell}_{\pm} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{-id_{\pm}}{2\alpha})}{\Gamma(\frac{1}{4}+\frac{2-id_{\pm}}{4\alpha})\Gamma(\frac{1}{4}-\frac{2+id_{\pm}}{4\alpha})},\quad b^{\ell}_{\pm} = i\,\frac{\Gamma(\frac{3}{2})\Gamma(\frac{-id_{\pm}}{2\alpha})}{\Gamma(\frac{3}{4}+\frac{2-id_{\pm}}{4\alpha})\Gamma(\frac{3}{4}-\frac{2+id_{\pm}}{4\alpha})}\,. \tag{24}$$

SUMMARY OF THE ANALYSIS

1) $\ell = 0$ mode.

For $\gamma < 5/3$, the limits of $\mathcal{F}_+/f$ and $\mathcal{G}_+/f$ are finite.

If $\gamma < 5/3$, $\mathcal{F}_-/f$ and $\mathcal{G}_-/f$ diverge asymptotically, which signals instability.

2) The Q_+^ℓ modes with $\ell > 1$.

In the asymptotic limit, both $\mathcal{F}_+/f$ and $\mathcal{G}_+/f$ diverge as f^κ as $f \to 0$, where κ is real and negative.

3) The asymptotic limits of both $\mathcal{F}_-/f$ and $\mathcal{G}_-/f$ of Q_-^ℓ diverge for $\ell > 1$ and $\gamma < 5/3$.

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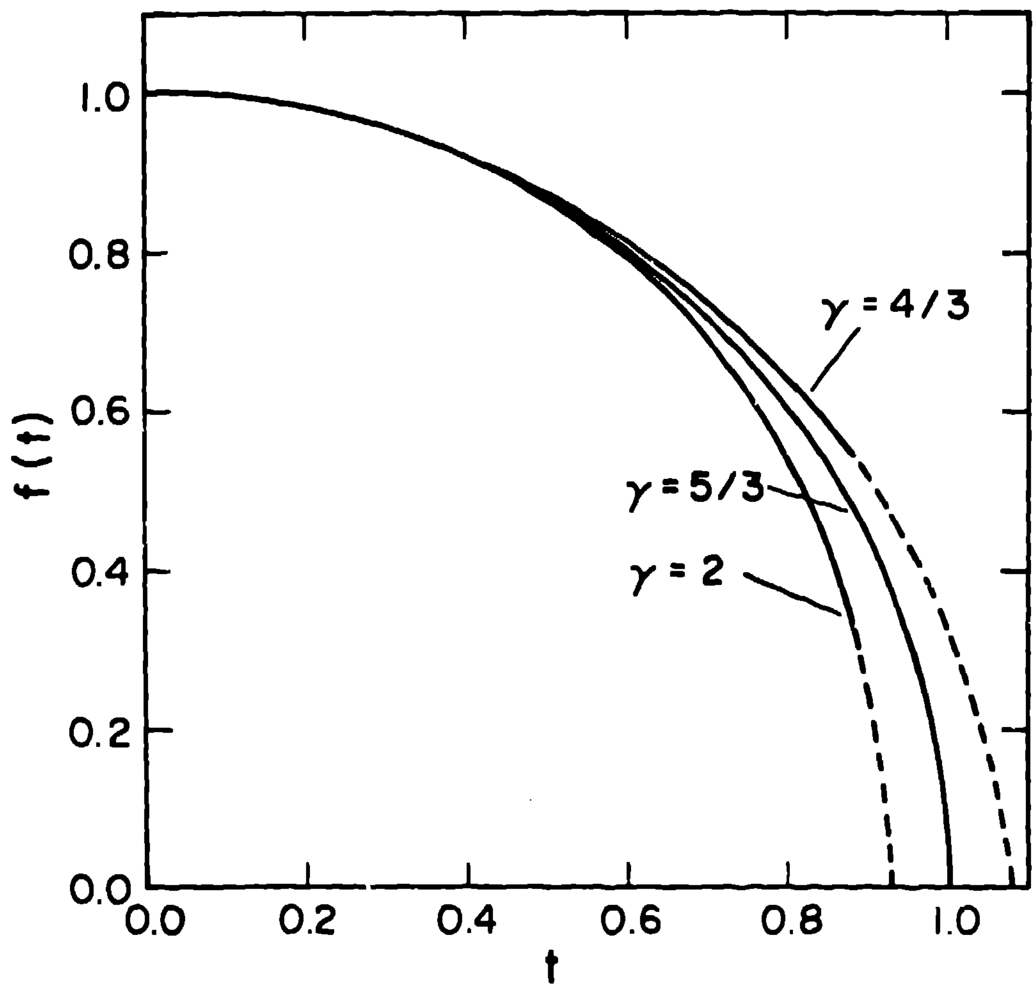
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f(t)
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1.0
t
$\gamma = 4/3$
$\gamma = 5/3$
$\gamma = 2$

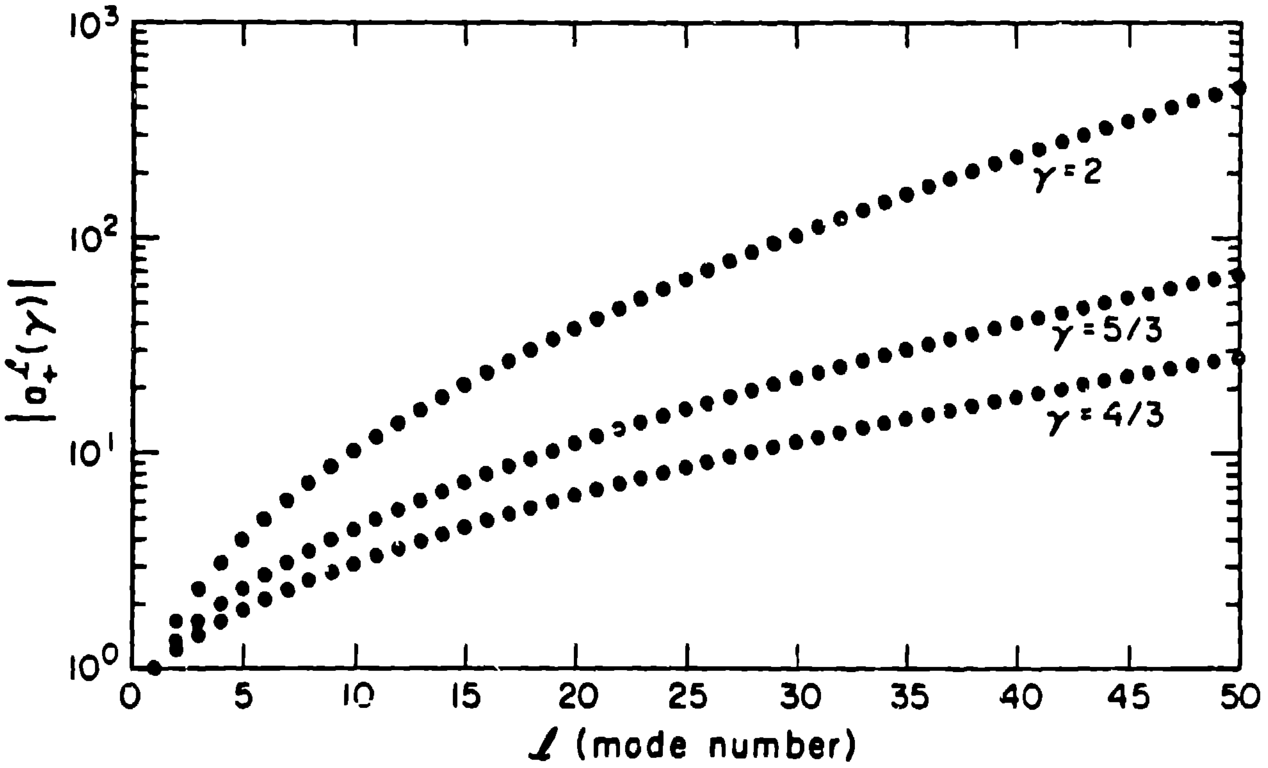

Fig. 1. The absolute values of $a_+^\ell(\gamma)$ given in Eq. (20) are plotted against the mode number ℓ = 1 to 50.

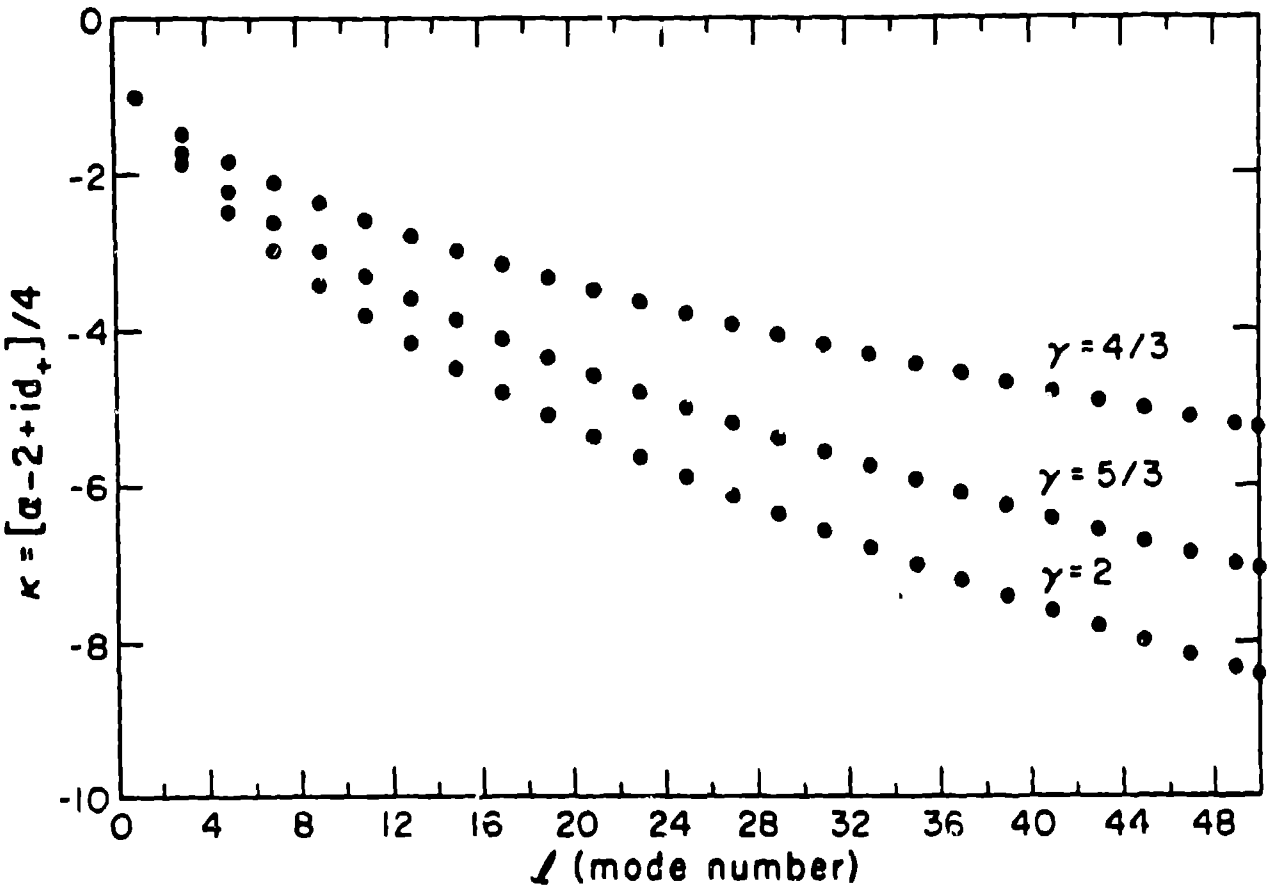

Fig. 2. The exponent of the divergence factor κ is plotted against the mode number.

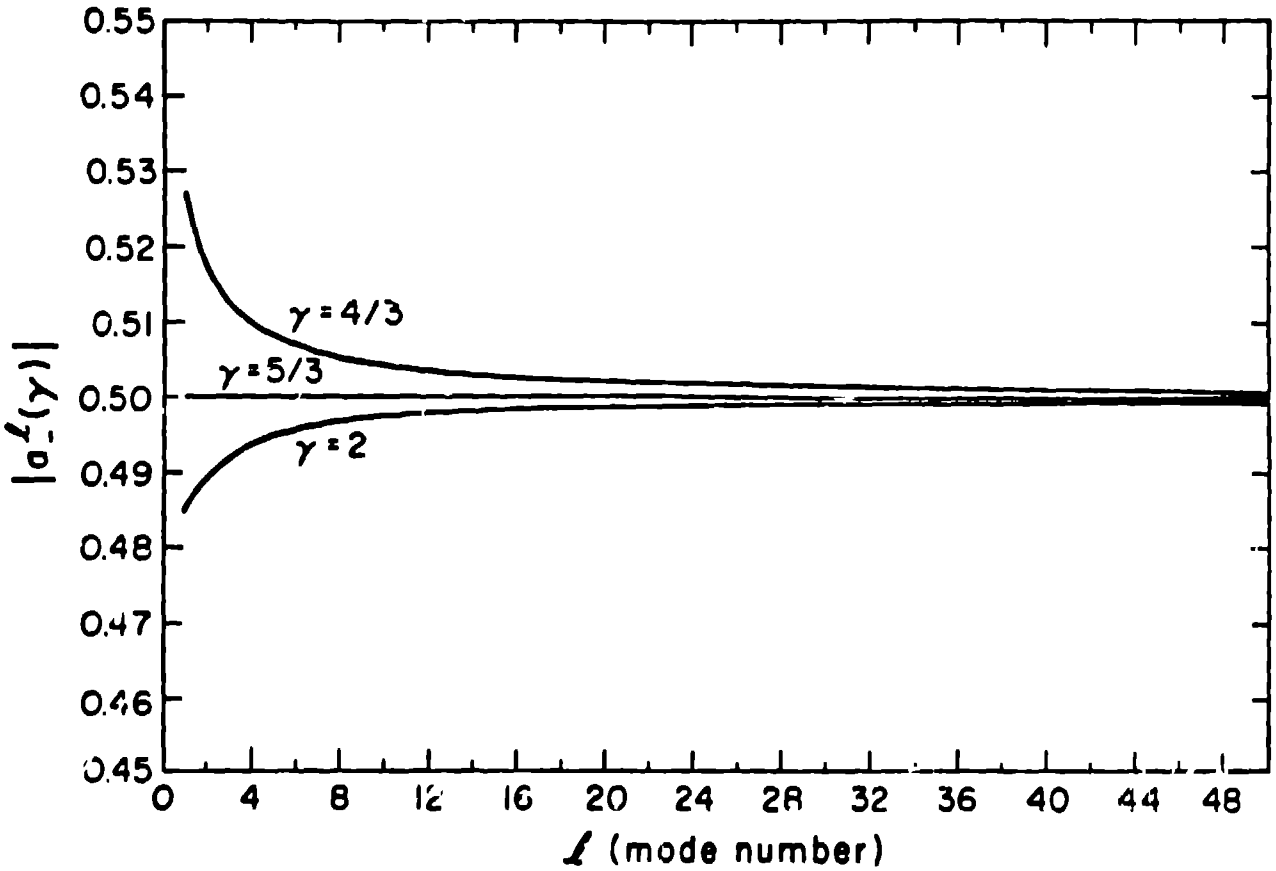

Fig. 3. The absolute values of $a_-^{\ell}(\gamma)$ for $\gamma = 4/3$, 5/3, and 2 are plotted against the mode number $\ell = 1$ to 50.

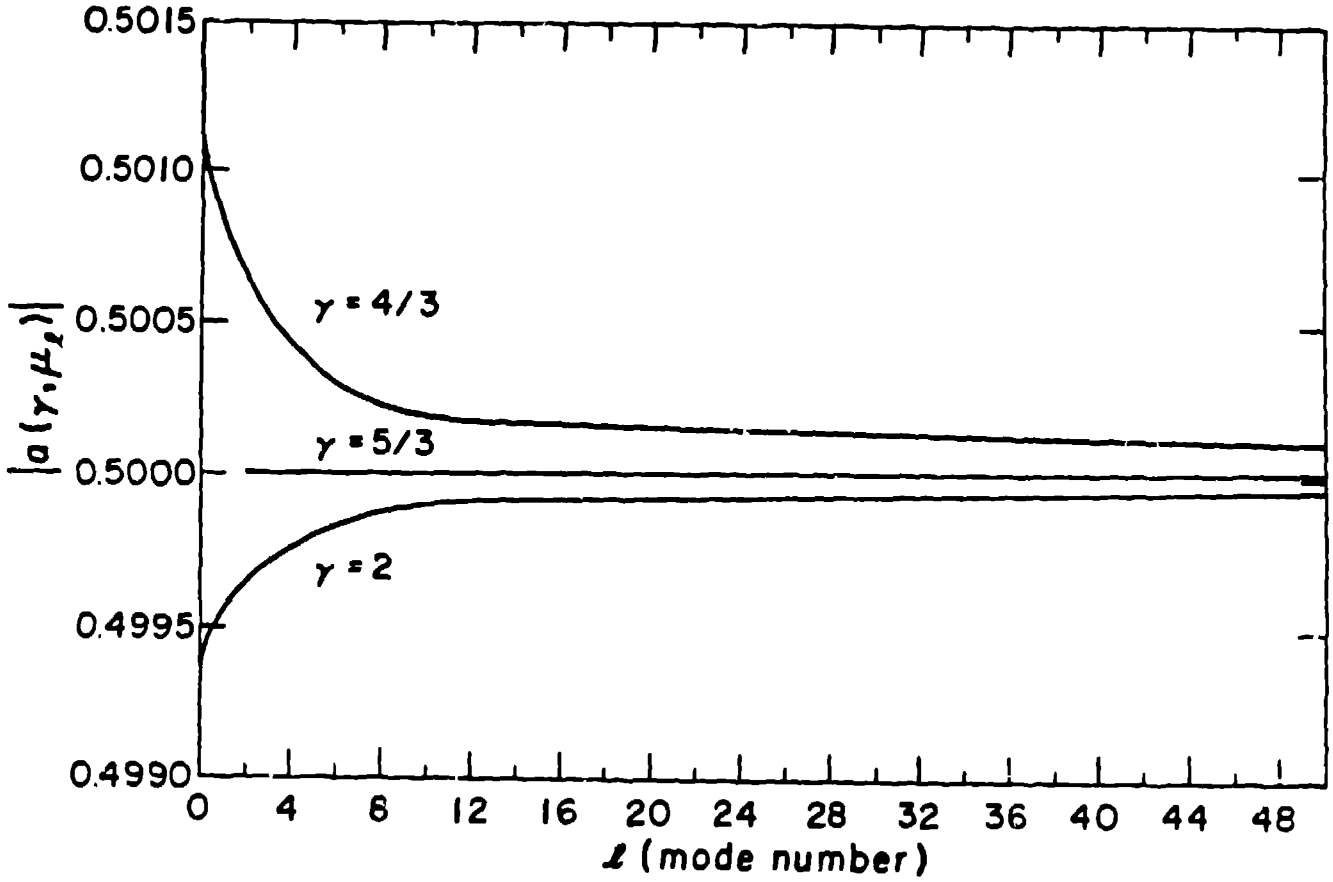

Fig. 4. The absolute values of $a(\gamma,\mu_\ell)$ for $\gamma = 4/3$, 5/3, and 2 are plotted against the mode number ℓ. Here μ_ℓ is the smallest eigenvalue for a given ℓ with $r_+ = 7.0$ cm and $r_- = 5.0$ cm.